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LETTER TO THE EDITOR

Internal Arnold diffusion and chaos thresholds in coupled symplectic maps

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Abstract. We investigate numerically how typical trajectories fill the phase space in low-dimensional symplectic (Hamiltonian) maps with finite phase space. We do not find any sign of a 'chaos threshold' as reported by other authors when the non-linearity parameters are increased. Instead, as expected from Arnold diffusion, we find that single trajectories fill most (if not all) of the coarse-grained phase space even for very small non-linearities. Due to the 'stickiness' of tori also observed in two-dimensional maps, this filling is much slower than what one might expect naively and is possibly described by power laws. The 'chaos threshold' observed in a previous paper is explained as a trivial effect.

It is well known that non-integrable Hamiltonian systems with two degrees of freedom (having two-dimensional Poincaré plots) are in general not ergodic [1]. Phase space separates there into chaotic domains and into tori on which the motion is regular. Even though the chaotic orbits are dense, a trajectory cannot penetrate through a torus and thus remains restricted to the regions of phase space accessible from the initial point.

This is no longer so for systems with more than two degrees of freedom. There, one expects also that chaotic regions and tori both should have finite measure. Hence, a random initial point should have finite probabilities both for staying on a torus and for moving chaotically. But there, tori have dimensions of at most half of the dimensionality of phase space, and they cannot separate phase space topologically. This *a priori* chance of a chaotic trajectory to access all of phase space is called Arnold diffusion.

In a system subject to external noise (not destroying symplecticity!), this Arnold diffusion should effectively destroy all regular motions: an infinitesimally small amount of noise is sufficient to push a regular trajectory off its torus. While this has no effect in systems with two degrees of freedom since the new chaotic trajectory will stay squeezed between other nearby tori, it can have a big effect in higher dimensions. One might conjecture that the same effect is played by round-off errors in computer simulations, but this is not at all clear. The main reason is that these errors are not random (even if the programmer cannot control them), but fully deterministic. Thus, they could have the opposite effect of forcing a trajectory into a periodic orbit, suppressing thereby the diffusion.

It is well known that Arnold diffusion is very slow. The reason is the 'stickiness' of tori: if a chaotic trajectory comes close to the boundary of the chaotic domain, it behaves essentially regularly and has to follow that boundary. The reason is simply that on the boundary the Lyapunov exponents are all zero, and thus the trajectory cannot diverge from the regular trajectory on the boundary nearest to it. This is very clearly seen in simulations of 2D maps like, e.g., the standard map

$$\begin{aligned} p' &= p - k \sin x \qquad && \bmod 2\pi \\ x' &= x + p' && \bmod 2\pi. \end{aligned} \tag{1}$$

In these systems, the stickiness is even enhanced by 'cantori' which are invariant sets resulting from tori which have just broken up. They act as very efficient barriers [2].

A priori one might also guess that these effects are smaller in higher dimensions, thus leading to increasingly larger chaotic regions and smaller timescales as dimensionality increases.

This is not exactly what is reported in a large number of papers [3-6]. It was found there that there are sharp thresholds in the non-linearity parameters (energy in flows, coupling constants in maps) such that the system is ergodic above the threshold and strongly non-ergodic below. This was found in flows [3-5] (Fermi-Pasta-Ulam α and β models, coupled Lennard-Jones potentials) as well as in maps [6], in the limit of large systems.

In contrast to this, we found in recent simulations [7] that there was no large-scale breaking of ergodicity for coupled standard maps

$$\begin{aligned} p_n' &= p_n + k \sin x_n + \beta[\sin(x_{n+1} - x_n) + \sin(x_{n-1} - x_n)] \qquad && \bmod 2\pi \\ x_n' &= x_n + p_n' \end{aligned} \tag{2}$$

with $n = 1, \ldots, N$, and with $x_{N+1} = x_1$. In these simulations, we computed the effective Lyapunov exponents by following the map and the tangent map over typically $\sim 10^7$-10^8 iterations and estimating the Lyapunov exponents from $T = 16$ to $T = 8192$ successive iterations. Starting values were chosen at random. For instance, for $k = 1.48$, $\beta = 0.25$ and $N = 3$, the same Lyapunov exponent spectrum was found for $\sim 99.8\%$ of all trajectories. It was only a small fraction ($\sim 2 \times 10^{-3}$) of the trajectories which seemed to stick to regular tori. Similar results were found for other non-linearity parameters and for larger values of N.

It was found at the same time that there are very long timescales involved, as the fluctuations in the Lyapunov spectrum decreased only very slowly with T. Typically, these fluctuations did not decrease $\sim 1/\sqrt{T}$ as expected naively from a central limit behaviour, but as a smaller power of T. Results are reproduced in figure 1. On the basis of the speed of this convergence, we could not rule out that even the few 'atypical' trajectories will finally become ergodic.

This suggests strongly that the system is ergodic nearly everywhere, at least in the version implemented on our computer. As we have said before, round-off errors might *a priori* have a big effect. One effect of round-off errors could be the levelling off of the curves in figure 1 at large T. The bulk of our simulations were done with 64-bit accuracy (on a CYBER 205 and on a MICRO-VAX). Test runs with 120-bit and 60-bit accuracy on a CYBER 175 showed no systematic effect except possibly for large non-linearities, although individual trajectories are of course different on different machines. Thus it seems that the accuracy of our simulation is indeed sufficient. Systematic deviations were only seen in test runs with 32-bit arithmetic (see below).

Unfortunately, the above simulations do not really prove that the system becomes ergodic since different parts in phase space could by chance have the same Lyapunov exponents. In order to test the ergodicity directly, and also to see Arnold diffusion directly, we did the following.

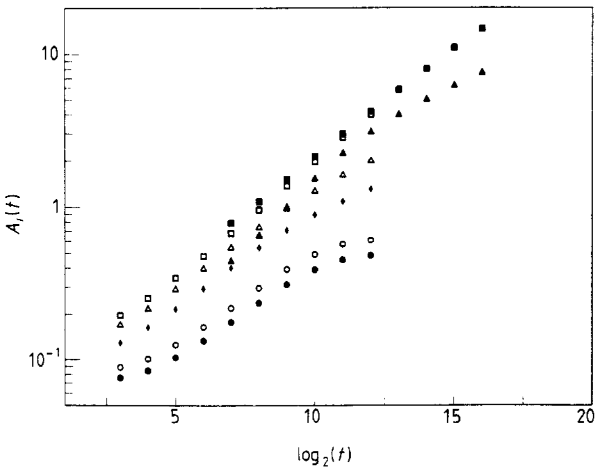

Figure 1. Fluctuations of effective Lyapunov exponents for (2) with $k = 1.02$, $\beta = 0.25$ and different values of N and i: ○, $N = 7$, $i = 4$; □, $N = 3$, $i = 2$; △ $N = 4$, $i = 2$; ◇, $N = 5$, $i = 3$; ●, $N = 8$, $i = 5$; ■, $N = 2$, $i = 1$; ▲, $N = 2$, $i = 2$. The quantity $A_i(T)$, defined in [7], is the mean square deviation of λ_i as estimated from a finite trajectory (length T) from the true λ_i, multiplied by T. It would be constant if there were no long-range effects due to 'sticky' tori and cantori.

We divided phase space into M^{2N} boxes by dividing every x_n and p_n interval into M bins each. This was done for $N = 2$ with M up to 46, and for $N = 3$ with M up to 12. Unfortunately, we could not do this for larger N, without going to unreasonably coarse grids. The results of these runs are summarised in figures 2–4.

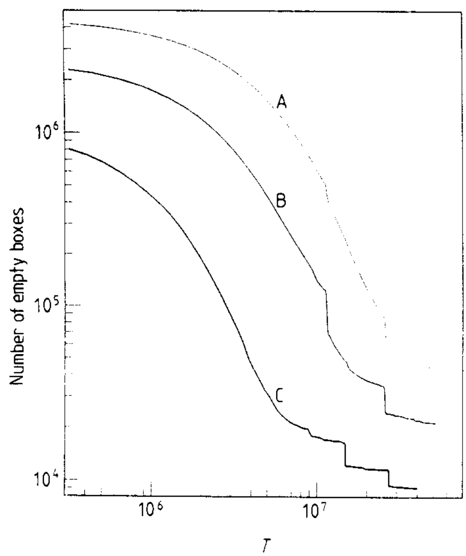

Figure 2. Number of boxes in hypercubic grids of a 4D phase space not yet visited by typical single trajectories of (2) with $N = 2$, $k = 0.8$ and $\beta = 0.2$, after T iterations with 120-bit arithmetic: A, 46^4 boxes; B, 40^4 boxes; C, 32^4 boxes.

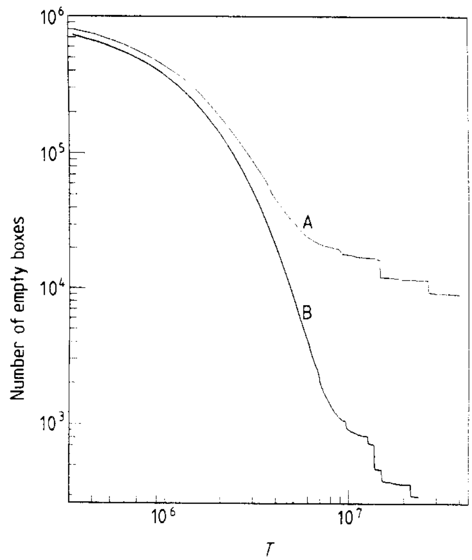

Figure 3. Number of unvisited boxes in hypercubic grids with (A) $N = 2$ and (B) $N = 3$. Parameters in both cases are $k = 0.8$, $\beta = 0.2$; mesh sizes are (A) 32^4, (B) 10^6.

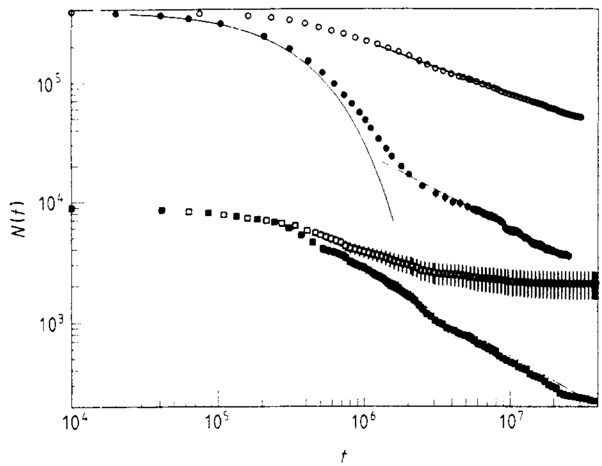

Figure 4. Average numbers of unvisited boxes in 4D phase space ($N = 2$), obtained by averaging over 30 trajectories each. ○: $k = 0.3$, $\beta = 0.1$, 25^4 boxes; ●: $k = 0.8$, $\beta = 0.2$, 25^4 boxes; ■: $k = 0.3$, $\beta = 0.1$, 10^4 boxes; □: $k = 0.3$, $\beta = 0.1$, 10^4 boxes, half precesion (32 bits). The straight lines are power law fits. The smooth curve is the prediction of a random jump process.

First, we show in figure 2 typical results from the same single trajectory with $N = 2$, $k = 0.8$ and $\beta = 0.2$. We see that this trajectory fills phase space quite efficiently at all levels of coarse graining. For short times, the filling is as one would expect from a random map: the fraction of unvisited boxes decreases exponentially, with the rate being just 1/(total number of boxes). But for larger T the filling slows down since new areas are filled only in very rare bursts which remind one of the breaking through cantori in 2D maps. These simulations were done with 120-bit words. To check for the possible effects of round-off errors, we also made simulations with 60-bit words (not shown). No significant systematic change was observed (though individual trajectories depend of course strongly on the round-off).

Results for $N = 2$ are compared in figure 3 with those for $N = 3$, at roughly the same number of boxes. Although the non-linearity parameters there are very small, we see that ergodicity is clearly not more broken for $N = 3$ than for $N = 2$. The opposite is true, as we had expected from the greater ease for Arnold diffusion.

In figure 4, we show results averaged over many (30) independent trajectories for the same set of parameters. These data are compatible with the asymptotic number of unvisited boxes decaying with a power law

$$N_{\text{empty}}(T) \sim \text{constant} \times T^{-\alpha}. \qquad (3)$$

The exponent α seems to depend both on the coarseness of the grid and on the parameters of the map ($\alpha = -0.63$ for $k = 0.8$, $\beta = 0.2$ and 25^4 boxes; $\alpha = -0.43$ (resp -0.60) for $k = 0.3$, $\beta = 0.1$, 25^4 (resp 10^4) boxes). This is in contrast to previous power laws suggested for symplectic maps [8, 9] which typically involve universal exponents. To estimate the effect of round-off errors, we repeated the last set of runs with half precision (32 bit instead of 64 bit). This time, we found a clear effect, also shown in figure 4, that reducing the precision suppresses Arnold diffusion. The reason for this seems to be that some trajectories are trapped into periodic orbits when working with too few digits. Anyhow, we see clearly that round-off errors cannot have enhanced Arnold diffusion as one might have feared.

Very similar in spirit to our box filling is the investigation by Farmer and Umberger [8] who observed how the number of visited boxes (for $T \to \infty$) in the standard map scales with the grid size $\varepsilon = 1/M$. From the dependence on ε they concluded that the connected chaotic region is a fat fractal. We cannot do the same here since the number of visited boxes does not seem to converge. Alternatively, some authors claimed to have found fractal trajectories [10], but it seems they have been misled by the slowness of Arnold diffusion [11].

These results strongly suggest that ergodicity holds down to much smaller non-linearities than studied in [7]. In order to test this we computed Lyapunov exponents for trajectories with random starting points and also for $N=2$, $k=0.8$ and $\beta=0.2$. The distributions of the largest Lyapunov exponents from 34 500 runs with $T=51\,200$ and from 69 000 runs with $T=25\,600$ are shown in figure 5. The integrals over the peaks near $\lambda=0$ (up to $\lambda \leqslant 0.15$) are 0.0115 and 0.0119, respectively. Given the long timescales involved, this indicates that for small non-linearity the breaking of ergodicity is indeed as small as suggested by box counting.

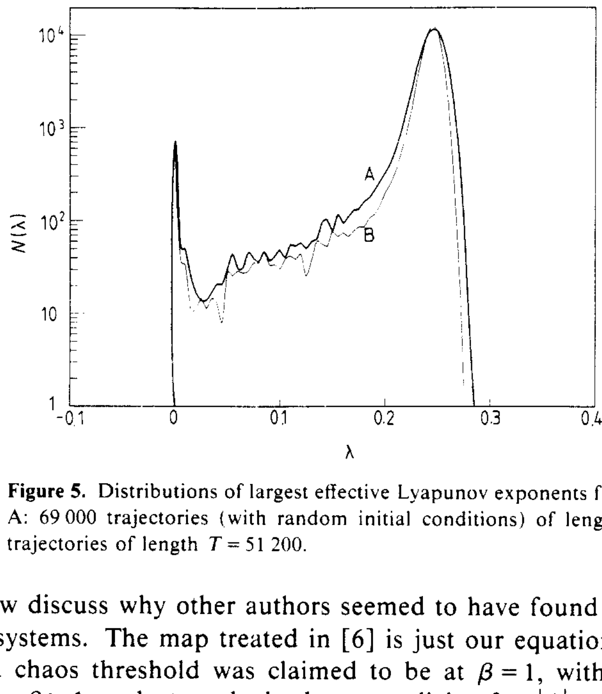

Figure 5. Distributions of largest effective Lyapunov exponents for $N=2$, $k=0.8$, $\beta=0.2$. A: 69 000 trajectories (with random initial conditions) of length $T=25\,600$; B: 34 500 trajectories of length $T=51\,200$.

Let us now discuss why other authors seemed to have found chaos thresholds in very similar systems. The map treated in [6] is just our equation (2) with $k=0$. In that paper, a chaos threshold was claimed to be at $\beta=1$, with essentially ergodic behaviour for $\beta>1$ and strongly broken ergodicity for $|\beta|<1$. The breaking of ergodicity was deduced from the fact that trajectories starting as sine waves,

$$p_n = 0 \qquad x_n = b + a\sin(2\pi nj/N) \tag{4}$$

with j being a small integer, did not evolve chaotically but stayed periodic for $|\beta| \leqslant 1$ and for sufficiently large N.

If we set $k=0$ in (2), the total momentum $P=\Sigma_n p_n$ is conserved. In order to decouple this zero mode, we rewrite (2) with $k=0$ in terms of the relative coordinates and momenta $\xi_n = x_{n+1} - x_n$, $q_n = p_{n+1} - p_n$:

$$\left.\begin{aligned} q_n' &= q_n + \beta(\sin\xi_{n+1} + \sin\xi_{n-1} - 2\sin\xi_n) \\ \xi_n' &= \xi_n + q_n' \end{aligned}\right\} \mod 2\pi. \tag{5}$$

The smooth ansatz (4) for the initial conditions leads for sufficiently large N to a point very close to the fixed point $\{q_n, \xi_n\} = 0^{2N}$ of (5). Whether or not a trajectory starting near this fixed point will be chaotic depends on whether the point is hyperbolic or elliptic. A linear stability analysis with an ansatz $(q_n, \xi_n) = \boldsymbol{u} \exp(\mathrm{i}\phi n)$ gives the eigenvalues

$$\lambda_{\pm} = 1 - 2\beta \sin^2 \phi/2 \pm [4\beta \sin^2 \phi/2(\beta \sin^2 \phi/2 - 1)]^{1/2}. \tag{6}$$

If there were no round-off errors, we should take $\phi = \pi j/N$ according to the sine wave ansatz, and the fixed point would be stable for all considered β. But round-off errors induce a small component (amongst others) with $\phi = \pi$ for which $\lambda = -1$ exactly at $\beta = 1$. For $\beta > 1$, we predict thus a transition to chaos via local checkerboard patterns exactly as observed in the simulations of [6].

Therefore, what looked like a chaos threshold in [6] is instead simply a change of stability of a single fixed point which has very little effect on the global amount of ergodicity. In particular, the phase space volume of regular orbits is unmeasurably small both above and below $\beta = 1$.

The situation is less clear concerning chaos thresholds in chains of oscillators (as opposed to maps) [3-5]. In these cases, the above argument does not apply. On the other hand, with the usual leap-frog method [5] these systems also become effectively coupled maps very similar to (5) with $\beta \approx 0$, but with different non-linearities. Taking just the first two terms in the Taylor series of the non-linear terms, we can nevertheless compare these cases with our simulations after suitably rescaling p_i and x_i. We find that the chaos thresholds reported in [3-5] correspond to trajectories starting off very close to $q_i = \xi_i = 0$. Thus there is no contradiction with our present results. For example, it could be that these 'thresholds' are transient phenomena, a possibility also not ruled out by the authors.

We conclude that non-ergodicity in chains of N coupled non-linear symplectic maps is small except possibly for very small non-linearities, and decreases with the number N. This is in contrast to a previous paper whose finding of a chaos threshold for large N is explained trivially. It is not in contradiction to the findings of chaos thresholds in coupled oscillators, due to the very different parameters in these papers. Due to the inherently finite resolution of any partition and to the very long timescales involved, we cannot rule out that ergodicity is broken. But for non-linearities of order one this breaking is very small already for $N = 2$, and any single chaotic trajectory seems to be dense in phase space. The latter is just what one would have expected from Arnold diffusion, although Arnold diffusion is strongly non-Brownian. Rather than being Brownian, it seems characterised by anomalous scaling laws with non-universal exponents. At present, we have no theoretical explanation for this.

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